The 1856 Lemma of Cayley Revisited

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February 1, 2008

Abstract

The result of the classical invariant theory (CIT) commonly referred to as Lemma of Cayley is reviewed. Its analogue in the invariant theory of Killing tensors (ITKT) defined in pseudo-Riemannian spaces of constant curvature is formulated and proven. Illustrative examples are provided.

1   Introduction

In recent years the classical invariant theory (CIT) of homogeneous polynomials has reinvented itself once again through new aspects of the Lie group theory

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(notably, the generalizations of the moving frames method due to Fels and Olver [1, 2] and Kogan [3], see also the relevant references therein), the rise of the modern computer algebra and new applications in other areas of mathematics (see Hilbert [4] and Olver [5] for a complete review and related references). Thus, in their pioneering 2002 paper McLenaghan *et al* [6] successfully planted the underlying ideas of CIT into the fertile field of the (geometric) study of Killing tensors defined in pseudo-Riemannian manifolds of constant curvature, which ultimately bore the fruit of a new theory (see also [7, 9, 10, 11, 12, 13, 14, 15]). The resulting *invariant theory of Killing tensors (ITKT)* shares many of the same essential features with the original CIT. In light of the fact that "Mathematics is the study of analogies between analogies" [16], we wish to continue developing ITKT by establishing more analogies with CIT. As is well known, the *main object* of study in CIT is a vector space of homogeneous polynomials under the action of the general linear group (or its subgroups), while the *main problem* is that of the determination of the functions of the parameters of the vector space in question that remain fixed under the action of the group. These functions, called *invariants* (Sylvester is credited as the first to coin the term), are very useful in solving various classification problems. In this study the vector spaces of particular importance are the spaces of *binary forms*, or homogeneous polynomials of degree n in two variables, originally referred to by Cayley as *quantics*. Let $\mathcal{Q}^n(\mathbb{R}^2)$ denote the vector space of binary forms of degree n over the reals. Then the dimension d of the space is given by

$$d = \dim \mathcal{Q}^n(\mathbb{R}^2) = n + 1. \tag{1.1}$$

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The general form of an element $Q(x, y)$ of the vector space $\mathcal{Q}^n(\mathbb{R}^2)$ is determined by the following formula.

$$Q(x, y) = \sum_{i=0}^{n} \binom{n}{i} a_i x^{n-i} y^i, \quad (x, y) \in \mathbb{R}^2. \tag{1.2}$$

Note the arbitrary constants $a_0, \ldots, a_n$ represent the parameter space $\Sigma \simeq \mathbb{R}^{n+1}$ corresponding to $\mathcal{Q}^n(\mathbb{R}^2)$. The special linear group $SL(2, \mathbb{R})$ (for example) acts on the space $\mathcal{Q}^n(\mathbb{R}^2)$ by linear substitutions, which yield the corresponding transformation rules

$$\begin{aligned}
\tilde{a}_0 &= \tilde{a}_0(a_0, \ldots, a_n, \alpha, \beta, \gamma, \delta), \\
\tilde{a}_1 &= \tilde{a}_1(a_0, \ldots, a_n, \alpha, \beta, \gamma, \delta), \\
&\vdots \\
\tilde{a}_n &= \tilde{a}_n(a_0, \ldots, a_n, \alpha, \beta, \gamma, \delta),
\end{aligned} \tag{1.3}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, $\alpha\delta - \beta\gamma = 1$ are local coordinates that parametrize the group. Note $\dim SL(2, \mathbb{R}) = 3$. The formulas (1.3) can be derived explicitly [5]. The problem is now reduced to finding all of the *invariants* of the $SL(2, \mathbb{R})$ action on the space Σ, or the functions of $a_0, \ldots, a_n$ that remain unchanged under the transformations (1.3):

$$\mathcal{I} = F(\tilde{a}_0, \ldots, \tilde{a}_n) = F(a_0, \ldots, a_n). \tag{1.4}$$

Note that in the case of $SL(2, \mathbb{R})$ acting on the vector space the invariants appear to be of weight zero due to the condition $\alpha\delta - \beta\gamma = 1$. In order to describe the space of all $SL(2, \mathbb{R})$-invariants of the vector space $\mathcal{Q}^n(\mathbb{R}^2)$ one has to determine a set of the *fundamental invariants*, with the property that all other invariants are (analytic) functions of the fundamental invariants. The number of fundamental invariants can be determined by using the result of the Fundamental Theorem on Invariants of a regular Lie group action [5]:

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Theorem 1.1 *Let G be a Lie group acting regularly on an m-dimensional manifold X with s-dimensional orbits. Then, in a neighborhood N of each point $x_0 \in X$, there exist $m - s$ functionally independent G-invariants $\Delta_1, \ldots, \Delta_{m-s}$. Any other G-invariant $\mathcal{I}$ defined near x_0 can be locally uniquely expressed as an analytic function of the fundamental invariants through $\mathcal{I} = F(\Delta_1, \ldots, \Delta_{m-s})$.*

The following proposition [5] provides a mechanism for determining the dimension of the orbits of a regular Lie group action.

Proposition 1.1 *Let a Lie group G act on X, $\mathfrak{g}$ is the Lie algebra of G and let $x \in X$. The vector space $S|_x = \mathrm{Span}\{\mathbf{V}_i(x) \,|\, \mathbf{V}_i \in \mathfrak{g}\}$ spanned by all vector fields determined by the infinitesimal generators at x coincides with the tangent space to the orbit $\mathcal{O}_x$ of G that passes through x, so $S|_x = T\mathcal{O}_x|_x$. In particular, the dimension of $\mathcal{O}_x$ equals the dimension of $S|_x$.*

One way to determine the fundamental invariants is to use the infinitesimal generators of the Lie algebra of the group, by which we mean their counterparts in the parameter space Σ satisfying the same commutator relations as the generators defined in the original space. Thus, a function $F(a_0, \ldots, a_n)$ is an invariant iff it is annihilated by the generators of the Lie algebra defined in the parameter space Σ. Accordingly, the problem of the determination of a set of the fundamental invariants reduces to solving the corresponding system of PDEs defined by the generators. This is a short description of Sophus Lie's *method of the infinitesimal generators*, which can be used to compute the invariants. Another powerful method, about which we shall not dwell in this paper, is Élie Cartan's *method of moving frames*, which has been recently brought back to light [1, 2, 5, 19, 17, 18, 3, 9, 11]. Arthur Cayley's main contributions to the development of CIT appeared during

the period 1854-1878 in his famous "ten memoirs on quantics". Having introduced the notion of an abstract group, he was the first to recognize that the action of a Lie group on a vector space can be investigated by studying its " infinitesimal action", that is the corresponding Lie algebra. In spite of the fact that Cayley thought of this as of something pertinent only to the general linear group and its subgroups, his results in this area may be considered as a precursor to Sophus Lie's theory of abstract Lie groups that was developed later in the 19th century. More specifically, in his "second memoirs on quantics" [20] Arthur Cayley considers (in modern mathematical language) the problem of the determination of the action of the Lie group $SL(2, \mathbb{R})$ on the vector space $\mathcal{Q}^n(\mathbb{R}^2)$ in conjunction with the problem of computing the invariants. The main result is the subject of the following lemma (see Cayley [20] and Olver [5], p.213).

Lemma 1.1 (Cayley) *The action of $SL(2, \mathbb{R})$ on the space $\mathcal{Q}^n(\mathbb{R}^2)$ of binary homogeneous polynomials of degree n defined by (1.2) has the following infinitesimal generators in the corresponding parameter space Σ:*

$$
\begin{aligned}
\mathbf{V}^- &= na_1\partial_{a_0} + (n-1)a_2\partial_{a_1} + \cdots + 2a_{n-1}\partial_{a_{n-2}} + a_n\partial_{a_{n-1}}, \\
\mathbf{V}^0 &= -na_0\partial_{a_0} + (2-n)a_1\partial_{a_1} + \cdots + (n-2)a_{n-1}\partial_{a_{n-1}} \\
&\quad + na_n\partial_{a_n}, \\
\mathbf{V}^+ &= a_0\partial_{a_1} + 2a_1\partial_{a_2} + \cdots + (n-1)a_{n-2}\partial_{a_{n-1}} + na_{n-1}\partial_{a_n},
\end{aligned}
\tag{1.5}
$$

where $\partial_{a_i} = \frac{\partial}{\partial a_i}$, $i = 0, \ldots, n$.

Observe that the vector fields (1.5) enjoy the following commutator relations

$$
[\mathbf{V}^-, \mathbf{V}^0] = -2\mathbf{V}^-, \quad [\mathbf{V}^+, \mathbf{V}^0] = 2\mathbf{V}^+, \quad [\mathbf{V}^-, \mathbf{V}^+] = \mathbf{V}^0, \tag{1.6}
$$

which confirm that the generators (1.5) represent the action of $SL(2, \mathbb{R})$ in the parameter space Σ. In view of the above, solving the problem of the determination of

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the $SL(2, \mathbb{R})$-invariants of the vector space $\mathcal{Q}^n(\mathbb{R}^2)$ for a specific n amounts now to solving the corresponding system of linear PDEs determined by the generators (1.5):

$$
\begin{aligned}
\mathbf{V}^-(F) &= 0, \\
\mathbf{V}^0(F) &= 0, \\
\mathbf{V}^+(F) &= 0.
\end{aligned}
\tag{1.7}
$$

for a (analytic) function F defined in the parameter space Σ. We note that according to Proposition 1.1 the dimension of the orbits of the $SL(2, \mathbb{R})$ action on $\mathcal{Q}^n(\mathbb{R}^2)$ can (locally) be determined by the number of linearly independent vector fields (1.5). Accordingly, by Theorem 1.1 the number of fundamental $SL(2, \mathbb{R})$-invariants is $n + 1 - s$, where $s \leq 3$ is the dimension of the orbits. Therefore for each particular n the general solution to the system (1.7) will take the form

$$
\mathcal{I} = F(\Delta_1, \Delta_2, \ldots, \Delta_\ell),
\tag{1.8}
$$

where $\Delta_i = \Delta_i(a_0, \ldots, a_n)$, $i = 1, \ldots, \ell$, $\ell = n + 1 - s$ are the fundamental $SL(2, \mathbb{R})$-invariants. To illustrate the procedure, let us recall the following well-known example [5].

Example 1.1 Consider the vector space $\mathcal{Q}^2(\mathbb{R}^2)$. The elements of $\mathcal{Q}^2(\mathbb{R}^2)$ enjoy the following general form.

$$
Q(x, y) = a_0 x^2 + 2a_1 xy + a_2 y^2.
\tag{1.9}
$$

The (local) action of $SL(2, \mathbb{R})$ in the parameter space $\Sigma \simeq \mathbb{R}^3$ generated by the parameters a_0, a_1 and a_2 is represented by the vector fields

$$
\begin{aligned}
\mathbf{V}^- &= a_0 \partial_{a_1} + 2a_1 \partial_{a_2}, \\
\mathbf{V}^0 &= 2a_0 \partial_{a_0} - 2a_2 \partial_{a_2}, \\
\mathbf{V}^+ &= 2a_1 \partial_{a_0} + a_2 \partial_{a_1}.
\end{aligned}
\tag{1.10}
$$

obtained via the standard technique of exponentiation. We immediately observe that only two vector fields (1.10) are linearly independent, therefore in view of Theorem 1.1 and Proposition 1.1 there is (almost everywhere) $3 - 2 = 1$ fundamental $SL(2, \mathbb{R})$-invariant of the vector space $\mathcal{Q}^2(\mathbb{R}^2)$. Indeed, solving the system of PDEs (1.7) for the vector fields (1.10) yields the solution:

$$\mathcal{I} = F(\Delta_1),$$

where $\Delta_1 = a_0 a_2 - a_1^2$. The group acts with orbits of two types: $a_0 = a_1 = a_2 = 0$, which is an orbit of dimension 0 and the level sets of Δ_1 (i.e., $\Delta_1 = 0$ and $\Delta_1 \neq 0$), both of which are orbits of dimension 2.

Now let us turn our attention to ITKT. Here the underlying space is a pseudo-Riemannian manifold $(M, \mathbf{g})$ of constant curvature. The vector spaces in question are the vector spaces of Killing tensors. Our notations are compatible with those introduced in [10]. Thus, for a fixed $n \geq 1$, $\mathcal{K}^n(M)$ denotes the vector space of Killing tensors of valence n defined on $(M, \mathbf{g})$. The group acting on $\mathcal{K}^n(M)$ is the isometry group $I(M)$ of $(M, \mathbf{g})$.

Remark 1.1 Here and below $I(M)$ denotes the *continuous* Lie group of isometries of M. We do not take into consideration discrete isometries.

A comprehensive review of ITKT is the subject of Section 2. Now, let us formulate an analogue of the problem solved by Cayley [20]. Since Cayley's problem concerns binary forms it will be natural to investigate in this respect the Killing tensors of arbitrary valence defined in pseudo-Riemannian manifolds of dimension two, for example, the Minkowski plane $\mathbb{R}_1^2$, More infromation about the Minkowski geometry can be found in Thompson [21]. Accordingly, the vector

spaces that we shall study in what follows are $\mathcal{K}^n(\mathbb{R}_1^2)$, $n \geq 1$. Table 1 presents a comparison of the "ingredients" and information that can be used to solve the two sister-problems. Having made these observations, we are now in the position to

Theory	Vector space	Group	Dimension of the space	Dimension of the orbits
CIT	$\mathcal{Q}^n(\mathbb{R}^2)$	$SL(2, \mathbb{R})$	$n + 1$	≤ 3
ITKT	$\mathcal{K}^n(\mathbb{R}_1^2)$	$I(\mathbb{R}_1^2)$	$\frac{1}{2}(n+1)(n+2)$	≤ 3

Table 1: The settings for the corresponding problems in CIT and ITKT.

formulate the ITKT version of the problem considered by Cayley in [20].

Problem 1 *Consider the action of the isometry Lie group $I(\mathbb{R}_1^2)$ on the vector space $\mathcal{K}^n(\mathbb{R}_1^2)$. Determine a representation of the corresponding Lie algebra $i(\mathbb{R}_1^2)$ on the parameter space Σ of $\mathcal{K}^n(\mathbb{R}_1^2)$.*

Clearly, the solution to this problem will mimic the result of Lemma 1.1, namely one will have to determine a basis of the Lie algebra defined on the parameter space Σ of $\mathcal{K}^n(\mathbb{R}_1^2)$, which is isomorphic to the Lie algebra $i(\mathbb{R}_1^2)$. Having the generators of such a Lie algebra will allow one to compute the $I(\mathbb{R}_1^2)$-invariants of $\mathcal{K}^n(\mathbb{R}_1^2)$ by solving the corresponding system of PDEs in the spirit of the corresponding problem of CIT described above. To solve the problem we need to establish first the requisite language of ITKT. This is the subject of the considerations that follow in Section 2.

2 Invariant theory of Killing tensors (ITKT)

Perhaps the most efficient way to begin describing a mathematical theory is by placing it among other mathematical theories. Recall that in the 19th century the post-"Theorema Egregium of Gauss" differential geometry branched off into two directions. Thus, B. Riemann [22] generalized the theory of surfaces of C. F. Gauss, from two to several dimensions, which ultimately led to the emergence of the new geometric objects known now as *(pseudo-) Riemannian manifolds*, and more broadly, today's *differential geometry*. The other school of thought was based on F. Klein's idea that every geometry could be interpreted as a theory of invariants with respect to a specific *transformation group*. Thus, according to F. Klein [23, 24], the main objective of any branch of geometry can be described as follows: "Given a manifold and a group of transformations of the manifold, to study the manifold configurations with respect to those features that are not altered by the transformations of the group" ([24], p.67). One of the most fundamental contributions of É. Cartan, in particular, with his *theory of moving frames* [25], is the fusion of these two directions into a single theory. The comprehensive monograph by Sharpe [26] unveils all of the beauty of Cartan's theory that subsumed the ideas of both Riemann and Klein (see also, for example, Arvanitoyeorgos [27]). The following diagram (see [26], p.ix) describes the relationship among the different approaches to geometry mentioned above.

$$
\begin{array}{ccc}
\text{Euclidean Geometry} & \xrightarrow{\text{generalization}} & \text{Klein Geometries} \\[1em]
\downarrow \text{generalization} & & \text{generalization} \downarrow \\[1em]
\text{Riemannian Geometry} & \xrightarrow{\text{generalization}} & \text{Cartan Geometries}
\end{array}
\qquad (2.11)
$$

ITKT can be placed into the theory of Cartan linking the developments of Riemann and Klein. We now shall present the evidence to justify this claim. Indeed, let $(M, \mathbf{g})$ be a pseudo-Riemannian manifold of constant curvature, assume also that $\dim M = m$.

Definition 2.1 *A* Killing tensor $\mathbf{K}$ *of valence* n *defined in* $(M, \mathbf{g})$ *is a symmetric* $(n, 0)$*-type tensor satisfying the Killing tensor equation*

$$[\mathbf{K}, \mathbf{g}] = 0, \tag{2.12}$$

where $[\ ,\ ]$ *denotes the Schouten bracket [28]. When* $n = 1$*,* $\mathbf{K}$ *is said to be a* Killing vector (infinitesimal isometry) *and the equation (2.12) reads*

$$\mathcal{L}_{\mathbf{K}}\mathbf{g} = 0,$$

where $\mathcal{L}$ *denotes the Lie derivative operator.*

Remark 2.1 Throughout this paper, unless otherwise specified, $[\ ,\ ]$ denotes the Schouten bracket, which is a generalization of the usual Lie bracket of vector fields.

The set of all Killing vectors of $(M, \mathbf{g})$, denoted by $i(M)$, is a Lie algebra of the corresponding Lie group of isometries $I(M)$, which is also a Lie subalgebra of the space $\mathcal{X}(M)$ of all vector fields defined on M. As is well-known, $d = \dim i(M) = \frac{1}{2}m(m + 1)$ iff the space $(M, \mathbf{g})$ is of constant curvature. It follows immediately from (2.12) that Killing tensors of the same valence n constitute a

vector space $\mathcal{K}^n(M)$. Moreover, the following properties hold true:

$$[\,,\,]\colon \mathcal{K}^n(M) \oplus \mathcal{K}^\ell(M) \to \mathcal{K}^{n+\ell-1}(M), \qquad (2.13)$$

$$[\mathbf{K}^n, \mathbf{K}^\ell] = -[\mathbf{K}^\ell, \mathbf{K}^n] \quad \text{(skew-symmetry)}, \qquad (2.14)$$

$$[[\mathbf{K}^n, \mathbf{K}^\ell], \mathbf{K}^r] + \text{(cycle)} = 0 \quad \text{(Jacobi identity)}, \qquad (2.15)$$

where $\mathbf{K}^n \in \mathcal{K}^n(M)$, $\mathbf{K}^\ell \in \mathcal{K}^\ell(M)$, $\mathbf{K}^r \in \mathcal{K}^r(M)$. Therefore one can consider a graded Lie algebra of Killing tensors defined on $(M, \mathbf{g})$ with respect to the Schouten bracket $[\,,\,]$:

$$\mathcal{K}_{alg} = \mathcal{K}^0(M) \oplus \mathcal{K}^1(M) \oplus \mathcal{K}^2(M) \oplus \cdots \oplus \mathcal{K}^n(M) \oplus \cdots, \qquad (2.16)$$

where $\mathcal{K}^0(M) = \mathbb{R}$, $\mathcal{K}^1(M) = i(M)$ and $n = 0, 1, 2, \ldots$, denotes the valence of the Killing tensors belonging to the corresponding space $\mathcal{K}^n(M)$. These remarkable geometrical objects have been actively studied for a long time by mathematicians and physicists alike. Apart from possessing beautiful mathematical properties, Killing tensors and conformal Killing tensors naturally arise in many problems of classical mechanics, general relativity, field theory and other areas. More information can be found, for example, in the following references: Delong [29], Dolan *et al* [30], Benenti [31], Bruce *et al* [18], Bolsinov and Matveev [32], Crampin [33], Eisenhart [34, 35], Fushchich and Nikitin [36], Kalnins [37, 38], Kalnins and Miller [39, 40], Miller [41], Mokhov and Ferapontov [42], Takeuchi [43], Thompson [44], as well as many others (more references related to the study of Killing tensors of valence two can be found in the review [31]). To illustrate how Killing tensors appear naturally in the problems of classical mechanics, let us consider the following example.

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Example 2.1 Let $(\mathbf{X}_H, \mathbf{P}_0, H)$ be a Hamiltonian system defined on $(M, \mathbf{g})$ by a natural Hamiltonian H of the form

$$H(\mathbf{q}, \mathbf{p}) = \frac{1}{2} g^{ij} p_i p_j + V(\mathbf{q}), \quad i, j = 1, \ldots, m, \tag{2.17}$$

where g^{ij} are the contravariant components of the corresponding metric tensor $\mathbf{g}$, $(\mathbf{q}, \mathbf{p}) \in T^*M$ are the canonical position-momenta coordinates and the Hamiltonian vector field $\mathbf{X}_H$ is given by

$$\mathbf{X}_H = [\mathbf{P}_0, H] \tag{2.18}$$

with respect to the canonical Poisson bi-vector $\mathbf{P}_0 = \sum_{i=1}^m \partial/\partial q^i \wedge \partial/\partial p_i$. Assume also that the Hamiltonian system defined by (2.17) admits a first integral of motion F which is a polynomial function of degree n in the momenta:

$$F(\mathbf{q}, \mathbf{p}) = K^{i_1 i_2 \ldots i_n}(\mathbf{q}) p_{i_1} p_{i_2} \ldots p_{i_n} + U(\mathbf{q}), \tag{2.19}$$

where $1 \leq i_1, \ldots, i_n \leq m$. Since the functions H and F are in involution, the vanishing of the Poisson bracket defined by $\mathbf{P}_0$:

$$\{H, F\}_0 = \mathbf{P}_0 \, \mathrm{d} H \, \mathrm{d} F = [[\mathbf{P}_0, H], F] = 0$$

yields

$$[\mathbf{K}, \mathbf{g}] = 0, \quad \text{(Killing tensor equation)} \tag{2.20}$$

and

$$K^{i_1 i_2 \ldots i_n} \frac{\partial V}{\partial q^{i_1}} p_{i_2} \ldots p_{i_n} = g^{ij} \frac{\partial U}{\partial q^i} p_j, \quad \text{(compatibility condition)}, \tag{2.21}$$

where the symmetric $(n, 0)$-tensor $\mathbf{K}$ has the components $K^{i_1 i_2 \ldots i_n}$ and $1 \leq i, j,$ $i_1, \ldots, i_n \leq m$. Clearly, in view of Definition 2.1 the equation (2.20) confirms

that $\mathbf{K}$ is a Killing tensor. Furthermore, in the case $n = 2$ (see Benenti [31]) the compatibility condition (2.21) reduces to $\mathbf{K} \, d V = \mathbf{g} \, d U$ or $d(\hat{\mathbf{K}} \, d V) = 0$, where the $(1,1)$-tensor $\hat{\mathbf{K}}$ is given by $\hat{\mathbf{K}} = \mathbf{K} \mathbf{g}^{-1}$.

Example 2.1 ellucidates the appearence of Killing tensors in the problems of the integrability theory of Hamiltonian systems. Notably, the geometric properties of Killing tensors of valence two have been routinely employed for a long time to solve the problems arising in the theory of orthogonal separation of variables (see, for example, [45, 46, 34, 35, 29, 41, 37, 38, 31, 18, 6, 12, 7] and the relevant references therein). Recall that the standard approach to the study of Killing tensors defined in pseudo-Riemannian spaces of constant curvature is based on the property that the Killing tensors defined in these spaces are sums of symmetrized tensor products of Killing vectors (see, for example, [29, 37, 44]).

Example 2.2 Consider the set $\mathcal{K}^2(\mathbb{R}_1^2)$ of all Killing tensors of valence two defined in $\mathbb{R}_1^2$ (Minkowski plane). Recall that the Lie algebra $i(\mathbb{R}_1^2)$ of Killing vectors (infinitesimal isometries) admits the basis given by the following Killing vectors:

$$\mathbf{T} = \partial_t, \quad \mathbf{X} = \partial_x, \quad \mathbf{H} = x\partial_t + t\partial_x \tag{2.22}$$

corresponding to t- and x-translations and (hyperbolic) rotation, given with respect to the standard pseudo-Cartesian coordinates (t, x). Note the generators (2.22) of the Lie algebra $i(\mathbb{R}_1^2)$ enjoy the following commutator relations:

$$[\mathbf{T}, \mathbf{X}] = 0, \quad [\mathbf{T}, \mathbf{H}] = \mathbf{X}, \quad [\mathbf{X}, \mathbf{H}] = \mathbf{T}. \tag{2.23}$$

Thus the general form of an element of $\mathcal{K}^2(\mathbb{R}_1^2)$ is given by

$$\begin{aligned}\mathbf{K} = {} & a_0 \mathbf{T} \odot \mathbf{T} + a_1 \mathbf{T} \odot \mathbf{X} + a_2 \mathbf{X} \odot \mathbf{X} + \\ & a_3 \mathbf{T} \odot \mathbf{H} + a_4 \mathbf{X} \odot \mathbf{H} + a_5 \mathbf{H} \odot \mathbf{H},\end{aligned} \tag{2.24}$$

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where $\odot$ stands for the symmetric tensor product and $a_0, \ldots, a_5 \in \mathbb{R}$ are arbitrary constants. The formula (2.24) can be used in the problem of classification of the elements of $\mathcal{K}^2(\mathbb{R}_1^2)$ and thus, the orthogonal coordinate webs that they generate. For more details, see Kalnins [37].

Another aproach that can be used in the study of Killing tensors of valence two is based on algebraic properties of the matrices that define this type of Killing tensors. Thus, in this case the problem of classification can be solved by making use of the eigenvalues and eigenvectors of the Killing tensors; for a complete description of the method see the review by Benenti [31] and the related references therein. These observations provide *compelling evidence that the study of Killing tensors lies within the framework of Riemann's approach to geometry.* Indeed, Killing tensors appear naturally in Riemann's metric geometry, as well as various physical models defined in terms of intrinsic geometry on pseudo-Riemannian spaces. A new approach to the study of Killing tensors introduced in [6] by McLenaghan, Smirnov and The is based on the fact that Killing tensors of a fixed valence defined on a pseudo-Riemannian manifold $(M, \mathbf{g})$ of constant curvature constitute a *vector space*. This easily follows from the $\mathbb{R}$-bilinear properties of the Schouten bracket [28] that appears in the fundamental formula (2.12). Accordingly, one can treat a Killing tensor as an element of its respective vector space.

Example 2.3 Consider again the *vector space* $\mathcal{K}^2(\mathbb{R}_1^2)$. Solving the Killing tensor equation (2.12) in the pseudo-Cartesian coordinates yields the general formula [12, 15]

$$
\mathbf{K} = \begin{pmatrix} a_0 + 2a_3x + a_5x^2 & a_1 + a_3t + a_4x + a_5tx \\ a_1 + a_3t + a_4x + a_5tx & a_2 + 2a_4t + a_5t^2 \end{pmatrix}, \tag{2.25}
$$

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of the elements of $\mathcal{K}^2(\mathbb{R}_1^2)$. The arbitrary constants of integration $a_0, \ldots, a_5$ are the same as in (2.24), they represent the dimension of the space $\mathcal{K}^2(\mathbb{R}_1^2)$. The formula (2.25) is the ITKT analgoue of the general formula (1.9), representing the elements of the vector space $\mathcal{Q}^2(\mathbb{R}^2)$ of quadratic forms in CIT.

We note that in the case of vector spaces of Killing tensors defined in $\mathbb{R}_1^2$, the generators (2.22) are not connected via any non-trivial relations. This is also true for any other two-dimensional pseudo-Riemannian manifold of constant curvature. In this view, for a fixed $n \geq 1$ the dimension of the corresponding vector space $\mathcal{K}^n(\mathbb{R}_1^2)$ can be computed, for example, by employing the well-known formula for the dimension of the space $\mathrm{Sym}^r(M)$ of symmetric $(r, 0)$-tensors defined over an m-dimensional manifold:

$$\dim \mathrm{Sym}^r(M) = \binom{m + r - 1}{r}. \qquad (2.26)$$

Indeed, in our case $m = \dim i(\mathbb{R}_1^2) = 3$ and $r = n$. Therefore we have from (2.26)

$$\dim \mathcal{K}^n(\mathbb{R}_1^2) = \frac{1}{2}(n + 1)(n + 2). \qquad (2.27)$$

For spaces of higher dimensions the formula (2.27) is no longer valid due to the existence of additional non-trivial relations among the generators of the Lie algebra of Killing vectors (i.e., the "syzygy modules problem" [29]). In the early 1980's the problem of extending the formula (2.27) to spaces of higher dimensions was solved independently and almost simultaneously by Delong [29], Takeuchi [43] and Thompson [44]. According to the *Delong-Takeuchi-Thompson (DTT) formula*, for a fixed $n \geq 1$ the dimension d of the vector space $\mathcal{K}^n(M)$ of Killing $(n, 0)$-tensors defined on an m-dimensional pseudo-Riemannian manifold $(M, \mathbf{g})$

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is given by

$$d = \dim \mathcal{K}^n(M) = \frac{1}{m}\binom{m+n}{n+1}\binom{m+n-1}{n}, \quad n \geq 1. \tag{2.28}$$

Note the formula (2.27) is in agreement with (2.28). Having the vector spaces of Killing tensors enables one to study them under the action of a *transformation group*. The most natural choice of such a group is, without any doubt, the corresponding Lie group of isometries $I(M)$ of the underlying pseudo-Riemannian manifold $(M, \mathbf{g})$. Indeed, it is easy to see that for a given vector space $\mathcal{K}^n(M)$, $n \geq 1$ defined on $(M, \mathbf{g})$ the corresponding isometry group $I(M)$ acts as an automorphism: $I(M) : \mathcal{K}^n(M) \to \mathcal{K}^n(M)$. This key observation made by McLenaghan *et al* [6] led to the emergence of ITKT. More specifically, the isometry group $I(M)$ acting on M induces the corresponding transformation laws on the parameters $a_0, \ldots, a_{d-1}$ of the vector space $\mathcal{K}^n(M)$:

$$\begin{aligned}
\tilde{a}_0 &= \tilde{a}_0(a_0, \ldots, a_{d-1}, g_1, \ldots, g_r), \\
\tilde{a}_1 &= \tilde{a}_1(a_0, \ldots, a_{d-1}, g_1, \ldots, g_r), \\
&\vdots \\
\tilde{a}_{d-1} &= \tilde{a}_{d-1}(a_0, \ldots, a_{d-1}, g_1, \ldots, g_r).
\end{aligned} \tag{2.29}$$

where $g_1, \ldots, g_r$ are local coordinates on $I(M)$ that parametrize the group, $r = \dim I(M) = \frac{1}{2}m(m+1)$ and d is given by (2.28). The formulas (2.29) can be obtained in each case by employing the standard transformation rules for tensors.

Example 2.4 Consider again the vector space $\mathcal{K}^2(\mathbb{R}_1^2)$. The corresponding isometry group $I(\mathbb{R}_1^2)$ acts in the Minkowski plane $\mathbb{R}_1^2$ parametrized by the standard pseudo-Cartesian coordinates (t, x) as follows.

$$\begin{pmatrix} \tilde{t} \\ \tilde{x} \end{pmatrix} = \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}, \tag{2.30}$$

where $\phi, a, b \in \mathbb{R}$ are local coordinates that parametrize the group $I(\mathbb{R}^2_1)$. We use the formula (2.30) and the transformation laws for $(2, 0)$ tensors

$$\tilde{K}^{ij}(\tilde{y}^1, \tilde{y}^2, \tilde{a}_0, \ldots, \tilde{a}_5) = K^{k\ell}(y^1, y^2, a_0, \ldots, a_5)\frac{\partial \tilde{y}^i}{\partial y^k}\frac{\partial \tilde{y}^j}{\partial y^\ell}, \quad i, j, k, \ell = 1, 2,$$

(2.31)

where the tensor components K^{ij} are given by (2.25), $y^1 = t, y^2 = x$, to obtain the transformation formulas for the parameters $a_0, \ldots, a_5$ that appear in (2.25) and define the parameter space of $\mathcal{K}^2(\mathbb{R}^2_1)$ [37, 12]:

$$
\begin{aligned}
\tilde{a}_0 &= a_0 \cosh^2 \phi + 2a_1 \cosh \phi \sinh \phi + a_2 \sinh^2 \phi + a_5 b^2 \\
&\quad -2(a_3 \cosh \phi + a_4 \sinh \phi)b, \\
\tilde{a}_1 &= a_1(\cosh^2 \phi + \sinh^2 \phi) + (a_0 + a_2) \cosh \phi \sinh \phi \\
&\quad -(aa_3 + ba_4) \cosh \phi - (aa_4 + ba_3) \sinh \phi + a_5 ab, \\
\tilde{a}_2 &= a_0 \sinh^2 \phi + 2a_1 \cosh \phi \sinh \phi + a_2 \cosh^2 \phi + a_5 a^2 \\
&\quad -2(a_4 \cosh \phi + a_3 \sinh \phi)a, \\
\tilde{a}_3 &= a_3 \cosh \phi + a_4 \sinh \phi - a_5 b, \\
\tilde{a}_4 &= a_3 \sinh \phi + a_4 \cosh \phi - a_5 a, \\
\tilde{a}_5 &= a_5.
\end{aligned}
$$

(2.32)

We note that the corresponding transformation formulas for the parameters obtained in [12] were derived for *covariant* Killing tensors. Accordingly, they differ somewhat from (2.32) presented above (compare with (7.6) in [12]). Clearly, the transformation formulas (2.32), and more generally - (2.29) are analogues of the corresponding transformation formulas in CIT (see, for example, (1.3)). It must be mentioned, however, that in the case of ITKT they are computationally more difficult to obtain. In view of the above observations, it is now easy to determine the ITKT analogue of the CIT-concept of an invariant.

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Definition 2.2 *Let $(M, \mathbf{g})$ be a pseudo-Riemannian manifold of constant curvature. For a fixed $n \geq 1$ consider the corresponding space $\mathcal{K}^n(M)$ of Killing tensors of valence n defined in $(M, \mathbf{g})$. A smooth function $\mathcal{I} : \Sigma \to \mathbb{R}$ defined in the space of functions over the parameter space Σ is said to be an $I(M)$-invariant of the vector space $\mathcal{K}^n(M)$ iff it satisfies the condition*

$$\mathcal{I} = F(a_0, \ldots, a_{d-1}) = F(\tilde{a}_o, \ldots, \tilde{a}_{d-1}) \tag{2.33}$$

under the transformation laws (2.29) induced by the isometry group $I(M)$.

We note that in a similar way the ITKT-analogues of the CIT-concepts of a *covariant* and *joint invariant* have been introduced in [9]. In complete analogy with CIT, we can in principle determine the space of $I(M)$-invariants for a specific vector space $\mathcal{K}^n(M)$, $n \geq 1$ by employing the (Sophus Lie) method of infinitesimal generators. To do so, one has to determine the (infinitesimal) action of $I(M)$ in the corresponding parameter space Σ of $\mathcal{K}^n(M)$ defined by the parameters $a_0, \ldots, a_{d-1}$. McLenaghan *et al* [6] devised an original procedure that can be used to derive the generators of the Lie algebra in Σ isomorphic to the Lie algebra $i(M)$ of $I(M)$ and thus, to compute the invariants. We briefly review the *MST-procedure* [6] here. Let $\{\mathbf{X}_1, \ldots, \mathbf{X}_r\}$ be the infinitesimal generators (Killing vector fields) of the Lie group $I(M)$ acting on M. Note Span $\{\mathbf{X}_1, \ldots, \mathbf{X}_r\} = \mathcal{K}^1(M) = i(M)$, where $i(M)$ is the Lie algebra of the Lie group $I(M)$. For a fixed $n \geq 1$, consider the corresponding vector space $\mathcal{K}^n(M)$. To determine the action of $I(M)$ in the parameter space Σ defined by $a_0, \ldots, a_{d-1}$, we find first the infinitesimal generators of $I(M)$ in Σ. Consider Diff Σ, it defines the corresponding space Diff $\mathcal{K}^n(M)$, whose elements are determined by the elements of Diff Σ in an ob-

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vious way. Let $\mathbf{K}^0 \in \mathrm{Diff}\, \mathcal{K}^n(M)$. Note $\mathbf{K}^0$ is determined by d parameters

$$a_i^0 = a_i^0(a_0, \ldots, a_{d-1}),$$

where $i = 0, \ldots, d-1$, which are functions of $a_0, \ldots, a_{d-1}$ - the parameters of Σ. Define now a map $\pi : \mathrm{Diff}\, \mathcal{K}^n(M) \to \mathcal{X}(\Sigma)$, given by

$$\mathbf{K}^0 \to \sum_{i=0}^{d-1} a_i^0(a_0, \ldots, a_{d-1})\partial_{a_i}. \tag{2.34}$$

To specify the action of $I(M)$ in Σ, we have to find the counterparts of the generators $\mathbf{X}_1, \ldots, \mathbf{X}_r$ in $\mathcal{X}(\Sigma)$. Consider the composition $\pi \circ \mathcal{L}$, where π is defined by (2.34) and $\mathcal{L}$ is the Lie derivative operator. Let $\mathbf{K}$ be the general Killing tensor of $\mathcal{K}^n(M)$, in other words $\mathbf{K}$ is the general solution to the Killing tensor equation (2.12). Next, define

$$\mathbf{V}_i = \pi \mathcal{L}_{\mathbf{X}_i} \mathbf{K}, \quad i = 1, \ldots r. \tag{2.35}$$

The composition map

$$\pi \circ \mathcal{L} : \, i(M) \to \mathcal{X}(\Sigma) \tag{2.36}$$

maps the generators $\mathbf{X}_1, \ldots, \mathbf{X}_r$ to $\mathcal{X}(\Sigma)$. Finally, we check that the vector fields $\mathbf{V}_i$, $i = 1, \ldots, r$ satisfy the same commutator relations as the original $\mathbf{X}_i$, $i = 1, \ldots, r$. This step is actually redundant, since it has been proven in general by showing that Killing tensors can be expressed as irreducible representations of $GL(n, \mathbb{R})$ that the vector fields (2.35) satisfy the same commutator relations as the original generators of $i(M)$ [8]. The main result of [8] is the proof of the corresponding conjecture formulated in [12]. Therefore we can use the vector fields (2.35) to solve the problem of the determination of the $I(M)$-invariants of the vectors space $\mathcal{K}^n(M)$ employing the (Sophus Lie) *method of infinitesimal*

generators by solving the corresponding system of (linear) PDEs generated by (2.35):

$$\mathbf{V}_i(F) = 0, \quad i = 1, \ldots, r. \tag{2.37}$$

The general solution to the system (2.37) describes the space of all $I(M)$-invariants of the vector space $\mathcal{K}^n(M)$, $n \geq 1$. The MST-procedure [6] based on *Lie derivative deformations of Killing tensors* is an analogue of the standard exponentiation used in CIT to determine the action of a group in the parameter space determined by the vector space of homogeneous polynomials in question. The technique of the Lie derivative deformations used here is a very powerful tool. It was used before, for example, in Smirnov [47] to generate compatible Poisson bi-vectors in the theory of bi-Hamiltonian systems. The idea introduced in [47] was used in [48] and applied to a different class of integrable systems. We also note that the generators (2.37) can be alternatively determined from the parameter transformation laws (2.29) when the latter are available. However, it is increasingly difficult and often impossible to determine (2.35) using (2.29) for vector spaces of Killing tensors of higher valence or defined in pseudo-Riemannian manifolds of higher dimensions. In what follows, we employ the MST-procedure to prove the ITKT-analogue of Cayley's lemma that is Problem 1 formulated in the previous section. To illustrate the effectiveness of the MST-procedure, let us consider the following example.

Example 2.5 Consider again the vector space $\mathcal{K}^2(\mathbb{R}_1^2)$. The action of the isometry group $I(\mathbb{R}_1^2)$ on the corresponding parameter space Σ defined by $a_0, \ldots, a_5$ (see (2.25)) is determined by the formulas (2.32). In order to determine the infinitesimal action of $I(\mathbb{R}_1^2)$ in Σ, we employ the MST-procedure. Thus, using the general formula (2.25) in conjunction with (2.35), we derive the corresponding generators

$\mathbf{V}_i, i = 1, 2, 3$:

$$\begin{aligned}
\mathbf{V}_1 &= a_3 \partial_{a_1} + 2a_4 \partial_{a_2} + a_5 \partial_{a_4}, \\
\mathbf{V}_2 &= a_4 \partial_{a_1} + 2a_3 \partial_{a_0} + a_5 \partial_{a_3}, \\
\mathbf{V}_3 &= -2a_1 \partial_{a_0} - a_4 \partial_{a_3} - (a_0 + a_2)\partial_{a_1} - 2a_1 \partial_{a_2} - a_3 \partial_{a_4}.
\end{aligned} \qquad (2.38)$$

We immediately note that the vector fields (2.38) satisfy the following commutator relations:

$$[\mathbf{V}_1, \mathbf{V}_2] = 0, \quad [\mathbf{V}_1, \mathbf{V}_3] = -\mathbf{V}_2, \quad [\mathbf{V}_2, \mathbf{V}_3] = -\mathbf{V}_1.$$

Choosing the basis $\{-\mathbf{V}_1, -\mathbf{V}_2, -\mathbf{V}_3\}$ reveals that the Lie algebra generated by (2.38) is isomorphic to the Lie algebra $i(\mathbb{R}_1^2) = \mathcal{K}^1(\mathbb{R}_1^2)$ generated by (2.22). Indeed, the vector fields (2.22) and $\{-\mathbf{V}_1, -\mathbf{V}_2, -\mathbf{V}_3\}$, where $\mathbf{V}_i, i = 1, 2, 3$ are given by (2.38) satisfy the same commutator relations (see (2.23)). We conclude therefore that the vector fields (2.38) represent the infinitesimal action of $I(\mathbb{R}_1^2)$ in Σ. Our next observation is that in view of Proposition 1.1 the orbits of the $I(\mathbb{R}_1^2)$-action have dimension three wherever the vector fields (2.38) are linearly independent. Therefore in that subspace of Σ, by Theorem 1.1, we expect to derive 6-3 = 3 fundamental $I(\mathbb{R}_1^2)$-invariants. The infinitesimal generators of the $I(\mathbb{R}_1^2)$-action in the 5-dimensional vector subspace of *non-trivial Killing two tensors* of $\mathcal{K}^2(\mathbb{R}_1^2)$ were determined in McLenaghan *et al* [13, 15].

Employing the method of characteristics to solve the system of PDEs (2.37) defined by the vector fields (2.38), we arrive at the following theorem.

Theorem 2.1 *Any algebraic $I(\mathbb{R}_1^2)$-invariant $\mathcal{I}$ of the subspace of the parameter space Σ of $\mathcal{K}^2(\mathbb{R}_1^2)$ defined by the condition that the vector fields (2.38) are linearly independent can be (locally) uniquely expressed as an analytic function*

$$\mathcal{I} = F(\Delta_1, \Delta_2, \Delta_3)$$

where the fundamental invariants Δ_i, $i = 1, 2, 3$ are given by

$$
\begin{aligned}
\Delta_1 &= a_5, \\
\Delta_2 &= (a_0 - a_2)a_5 - a_3^2 + a_4^2, \\
\Delta_3 &= (a_3^2 + a_4^2 - a_5(a_0 + a_2))^2 - 4(a_5 a_1 - a_3 a_4)^2.
\end{aligned}
\tag{2.39}
$$

The fact that $\Delta_1 = a_5$ is a fundamental $I(\mathbb{R}_1^2)$-invariant of the vector space $\mathcal{K}^2(\mathbb{R}_1^2)$ can be trivially deduced from the transformation formulas (2.32). The fundamental $I(\mathbb{R}_1^2)$-invariant Δ_3 presented above was first derived in McLenaghan *et al* [13, 15] and used to generate *discrete* $I(\mathbb{R}_1^2)$-invariants, which were in turn employed to classify orthogonal coordinate webs in the Minkowski plane $\mathbb{R}_1^2$. The same problem was solved in [9] by employing the $I(\mathbb{R}_1^2)$-invariants and covariants of the vector space $\mathcal{K}^2(\mathbb{R}_1^2)$. The observations and results summarized above put in evidence that *ITKT is a part of F. Klein's approach to geometry*. This is especially evident when one considers the vector spaces of Killing tensors of valence two. Thus, for example, in Horwood *et al* [7] the orthogonal coordinate webs of the Euclidean space $\mathbb{R}^3$ were completely classified in terms of the $I(\mathbb{R}^3)$-invariants. This is something to be expected since the theory of orthogonal coordinate webs of $\mathbb{R}^3$ is a part of the Euclidean geometry which, according to Felix Klein's "Erlangen Program" [23, 24], is an invariant theory of the corresponding isometry group $I(\mathbb{R}^3)$. In Section 3 we use the results presented above, in particular, the MST-procedure, to solve Problem 1.

3 The ITKT analogue of Cayley's lemma

In this section we prove the ITKT analogue of the Cayley Lemma [20] presented in Section 1. The vector space $\mathcal{K}^n(\mathbb{R}_1^2)$ appears to be a natural counterpart of the

vector space $\mathcal{Q}^n(\mathbb{R}^2)$ in CIT. The problem can be solved by employing the MST-procedure described in the previous section. To proceed, we need to derive first a general formula for the elements of $\mathcal{K}^n(\mathbb{R}_1^2)$ (i.e., an analogue of (1.2)). Note, that by (2.27) the dimension of the vector space in question is $(n+1)(n+2)/2$. Thus, each contravariant tensor $\mathbf{K} \in \mathcal{K}^n(\mathbb{R}_1^2)$ is determined by $(n+1)(n+2)/2$ parameters that appear in the $n+1$ components of the form

$$K^{i_1\dots i_p j_1\dots j_{n-p}}, \tag{3.40}$$

where $i_1 = \dots = i_p = 1$, $j_1 = \dots = j_{n-p} = 2$ and $p = 0, 1, \dots, n$. To derive the formulas for the components (3.40), we solve the Killing tensor equation (2.12) in the coordinates (t, x), which in this case reduces to the following system of PDEs:

$$\begin{cases} \partial_t K^{i_1\dots i_n} = 0, \\ (n-p+1)\partial_x K^{i_1\dots i_p j_1\dots j_{n-p}} = p\partial_t K^{i_1\dots i_{p-1} j_1\dots j_{n-p+1}}, \\ \partial_x K^{j_1\dots j_n} = 0, \end{cases} \tag{3.41}$$

where $p = 0, 1, \dots, n$, $\partial_t = \frac{\partial}{\partial t}$, $\partial_x = \frac{\partial}{\partial x}$. As a consequence of (3.41), we readily obtain the necessary differential conditions:

$$\begin{aligned} (\partial_x)^{p+1} K^{i_1\dots i_p j_1\dots j_{n-p}} &= 0, \\ (\partial_t)^{n-p+1} K^{i_1\dots i_p j_1\dots j_{n-p}} &= 0. \end{aligned} \tag{3.42}$$

Solving (3.42), we arrive at the following result. Each component of (3.40) is a mixed polynomial of degree p in x and degree q in t: :

$$K^{i_1\dots i_p j_1\dots j_q} = \begin{cases} \displaystyle\sum_{i=0}^{q} \binom{p}{i} t^i \sum_{j=0}^{p} \binom{p}{j} a_{pij} x^j, & \text{if} \quad n \geq p \geq \left[\frac{n+1}{2}\right], \\ \displaystyle\sum_{i=0}^{p} \binom{q}{i} x^i \sum_{j=0}^{q} \binom{q}{j} b_{qij} t^j, & \text{if} \quad 0 \leq p \leq \left[\frac{n+1}{2}\right], \end{cases} \tag{3.43}$$

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where $q = n - p$ and the parameters a_{pij}, b_{qij} are to be determined (at this stage they are inserted for mere convenience). We immediately recognize that the formula (3.43) is the ITKT analogue of the general formula (1.2) exhibited in Section 1. The parameters a_{pij}, b_{qij} can be determined by following the general procedure of solving the system of PDEs (3.41). For convenience we consider separately two cases: $n = 2k + 1$ and $n = 2k$. The parameters of each of the $n + 1$ components can be organized into groups in such a way that the parameters of one group are completely determined by the parameters of the other (see the illustrative examples below). After relabelling the parameters, we arrive at the following two schemes (corresponding to $n = 2k$ and $n = 2k + 1$ respectively), which specify the arrangements of the parameters of the first groups of the components. Once they are specified, the parameters of the other groups can be determined accordingly. **Case 1:** $n = 2k$

$$
\begin{array}{llllll}
\text{Step 1}: & a_0^1 & a_1^1 & \cdots & a_{n-2}^1 \quad a_{n-1}^1 \quad a_n^1, \\
& b_0^1 & b_1^1 & \cdots & b_{n-2}^1 \quad b_{n-1}^1 \quad a_n^1 \\[2mm]
\text{Step 2}: & a_0^2 & a_1^2 & \cdots & a_{n-3}^2 \quad a_{n-2}^2 \quad b_{n-1}^1 \\
& b_0^2 & b_1^2 & \cdots & b_{n-3}^2 \quad a_{n-2}^2 \quad a_{n-1}^1 \\[4mm]
& & & \vdots & \\[4mm]
\text{Step } \dfrac{n}{2}: & a_0^{\frac{n}{2}} \quad a_1^{\frac{n}{2}} \quad a_2^{\frac{n}{2}} \quad b_1^{\frac{n-2}{2}} & \cdots & b_{\frac{n+2}{2}}^1 \\
& b_0^{\frac{n}{2}} \quad b_1^{\frac{n}{2}} \quad a_2^{\frac{n}{2}} \quad a_1^{\frac{n-2}{2}} & \cdots & a_{\frac{n+2}{2}}^1 \\[4mm]
\text{Step} \dfrac{n+2}{2}: & a_0^{\frac{n+2}{2}} \quad b_1^{\frac{n}{2}} \quad b_2^{\frac{n-2}{2}} \quad b_3^{\frac{n-4}{2}} & \cdots & b_{\frac{n}{2}}^1
\end{array}
\tag{3.44}
$$

Case 2: $n = 2k + 1$

$$
\begin{array}{lllllll}
\text{Step 1}: & a_0^1 & a_1^1 & \cdots & a_{n-2}^1 & a_{n-1}^1 & a_n^1, \\
& b_0^1 & b_1^1 & \cdots & b_{n-2}^1 & b_{n-1}^1 & a_n^1
\end{array}
$$

$$
\begin{array}{lllllll}
\text{Step 2}: & a_0^2 & a_1^2 & \cdots & a_{n-3}^2 & a_{n-2}^2 & b_{n-1}^1 \\
& b_0^2 & b_1^2 & \cdots & b_{n-3}^2 & a_{n-2}^2 & a_{n-1}^1
\end{array}
$$

$$\vdots$$

$$
\begin{array}{llllllll}
\text{Step}\dfrac{n-1}{2}: & a_0^{\frac{n-1}{2}} & a_1^{\frac{n-1}{2}} & a_2^{\frac{n-1}{2}} & a_3^{\frac{n-1}{2}} & \cdots & b_{\frac{n+3}{2}}^1 \\[2mm]
& b_0^{\frac{n-1}{2}} & b_1^{\frac{n-1}{2}} & b_2^{\frac{n-1}{2}} & a_3^{\frac{n-1}{2}} & \cdots & a_{\frac{n+3}{2}}^1
\end{array}
$$

$$
\begin{array}{llllllll}
\text{Step}\dfrac{n+1}{2}: & a_0^{\frac{n+1}{2}} & a_1^{\frac{n+1}{2}} & b_2^{\frac{n-1}{2}} & b_3^{\frac{n-3}{2}} & \cdots & b_{\frac{n+1}{2}}^1 \\[2mm]
& b_0^{\frac{n+1}{2}} & a_1^{\frac{n+1}{2}} & a_2^{\frac{n-1}{2}} & a_3^{\frac{n-3}{2}} & \cdots & a_{\frac{n+1}{2}}^1
\end{array}
$$

$$(3.45)$$

The parameters that appear in the general solution to (3.43) are now organized in two schemes according the cases of n being even (3.45) and odd (3.44) respectively. More specifically, we first give $2(n+1) - 1$ parameters

$$a_0^1, \ldots, a_{n-1}^1, a_n^1 b_0^1, \ldots, b_{n-1}^1, a_n^1,$$

and then write down the first and the last components of the general element $\mathbf{K} \in \mathcal{K}^n(\mathbb{R}_1^2)$ as follows:

$$
K^{11\cdots 11} = \left[a_0^1 + \binom{n}{1} a_1^1 x + \binom{n}{2} a_2^1 x^2 + \ldots + \binom{n}{n-1} a_{n-1}^1 x^{n-1} + a_n^1 x^n \right],
$$

$$
K^{22\cdots 22} = \left[b_0^1 + \binom{n}{1} b_1^1 t + \binom{n}{2} b_2^1 t^2 + \ldots + \binom{n}{n-1} b_{n-1}^1 t^{n-1} + a_n^1 t^n \right].
$$

Next step: For $2(n-1) - 1$ new parameters

$$a_0^2, \ldots, a_{n-3}^2, a_{n-2}^2, b_0^2, \ldots, b_{n-3}^2, a_{n-2}^2$$

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we then write down the second and penultimate components of **K** as follows (see (3.43)), each of which is the sum of two polynomials, the first having been determined by the newly specified parameters and the other - by the parameters determined previously.

$$
K^{11\cdots12} = \left[a_0^2 + \binom{n-1}{1}a_1^2 x + \ldots + \binom{n-1}{n-2}a_{n-2}^2 x^{n-2} + b_{n-1}^1 x^{n-1} \right]
$$

$$
+ t\left[a_1^1 + \binom{n-1}{1}a_2^1 x + \ldots + \binom{n-1}{n-2}a_{n-1}^1 x^{n-2} + a_n^1 x^{n-1} \right],
$$

$$
K^{22\cdots21} = \left[b_0^2 + \binom{n-1}{1}b_1^2 t + \ldots + \binom{n-1}{n-2}a_{n-2}^2 t^{n-2} + a_{n-1}^1 t^{n-1} \right]
$$

$$
+ x\left[b_1^1 + \binom{n-1}{1}b_2^1 t + \ldots + \binom{n-1}{n-2}b_{n-1}^1 t^{n-2} + a_n^1 t^{n-1} \right].
$$

To clarify the process more, let us consider the next step (if any)for the given $2(n-3) - 1$ parameters

$$
a_0^3, a_1^3, \ldots, a_{n-5}^3, a_{n-4}^3, b_0^3, b_1^3, \ldots, b_{n-5}^3, a_{n-4}^3
$$

we write down the next two comonents as follows:

$$
K^{11\cdots122} =
$$

$$
\left[a_0^3 + \binom{n-2}{1}a_1^3 x + \ldots + \binom{n-2}{n-4}a_{n-4}^3 x^{n-4} + \binom{n-2}{n-3}b_{n-3}^2 x^{n-3} + b_{n-2}^1 x^{n-2} \right]
$$

$$
+ 2t\left[a_1^2 + \binom{n-2}{1}a_2^2 x + \ldots + \binom{n-2}{n-4}a_{n-3}^2 x^{n-4} + \binom{n-2}{n-3}a_{n-2}^2 x^{n-3} + b_{n-1}^1 x^{n-2} \right]
$$

$$
+ t^2\left[a_2^1 + \binom{n-2}{1}a_3^1 x + \ldots + \binom{n-2}{n-4}a_{n-2}^1 x^{n-4} + \binom{n-2}{n-3}a_{n-1}^1 x^{n-3} + a_n^1 x^{n-2} \right],
$$

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$$K^{22\cdots211} =$$

$$\left[b_0^3 + \binom{n-2}{1} b_1^3 t + \ldots + \binom{n-2}{n-4} a_{n-4}^3 t^{n-4} + \binom{n-2}{n-3} a_{n-3}^2 t^{n-3} + a_{n-2}^1 t^{n-2} \right]$$

$$+ 2x \left[b_1^2 + \binom{n-2}{1} b_2^2 t + \ldots + \binom{n-2}{n-4} b_{n-3}^2 t^{n-4} + \binom{n-2}{n-3} a_{n-2}^2 t^{n-3} + a_{n-1}^1 t^{n-2} \right]$$

$$+ x^2 \left[b_2^1 + \binom{n-2}{1} b_3^1 t + \ldots + \binom{n-2}{n-4} b_{n-2}^1 t^{n-4} + \binom{n-2}{n-3} b_{n-1}^1 t^{n-3} + a_n^1 t^{n-2} \right].$$

We repeat this process in both directions (i.e., going "downwards" and "upwards") until it is terminated in the middle of (3.43). In this view, counting the steps in both cases (n is even and n is odd), it is easy to see that the dimension of the space

$$d = \mathcal{K}^n(\mathbb{R}_1^2) = \frac{1}{2}(n+1)(n+2), \quad n \geq 1$$

gets decomposed as follows.

$$d = \begin{cases} [2(n+1)-1] + [2(n-1)-1] + \cdots + [2 \times 2 - 1] & \text{if } n \text{ is odd,} \\ [2(n+1)-1] + [2(n-1)-1] + \cdots + [2 \times 1 - 1] & \text{if } n \text{ is even.} \end{cases}$$
$$(3.46)$$

The auxiliary problem of finding the general form for the elements $\mathbf{K} \in \mathcal{K}^n(\mathbb{R}_1^2)$ is therefore completely solved. We immediately notice that the coefficients in the general solution (2.25) can be relabeled following the scheme (3.44) as follows: $a_0 = a_0^1$, $a_1 = a_0^2$, $a_2 = b_0^1$, $a_3 = a_1^1$, $a_4 = b_1^1$ and $a_5 = a_2^1$. To illustrate our results, let us consider more challenging examples.

Example 3.1 Consider the vector space $\mathcal{K}^4(\mathbb{R}_1^2)$, note $d = \dim \mathcal{K}^4(\mathbb{R}_1^2) = (4 + 1)(4 + 2)/2 = 15$. Following the coefficient scheme (3.44), we arrive at the

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following formulas for the components of the elements of $\mathcal{K}^4(\mathbb{R}_1^2)$.

$$
\begin{aligned}
K^{1111} &= a_0^1 + 4a_1^1 x + 6a_2^1 x^2 + 4a_3^1 x^3 + a_4^1 x^4, \\
K^{1112} &= (a_0^2 + 3a_1^2 x + 3a_2^2 x^2 + b_3^1 x^3) + t(a_1^1 + 3a_2^1 x + 3a_3^1 x^2 + a_4^1 x^3), \\
K^{1122} &= (a_0^3 + 2b_1^2 x + b_2^1 x^2) + 2t(a_1^2 + 2a_2^2 x + b_3^1 x^2) \\
&\quad + t^2(a_2^1 + 2a_3^1 x + a_4^1 x^2), \\
K^{1222} &= (b_0^2 + 3b_1^2 t + 3a_2^2 t^2 + a_3^1 t^3) + x(b_1^1 + 3b_2^1 t + 3b_3^1 t^2 + a_4^1 t^3), \\
K^{2222} &= b_0^1 + 4b_1^1 t + 6b_2^1 t^2 + 4b_3^1 t^3 + a_4^1 t^4.
\end{aligned}
$$

$$(3.47)$$

Example 3.2 Consider the vector space $\mathcal{K}^5(\mathbb{R}_1^2)$. In this case $d = \dim \mathcal{K}^5(\mathbb{R}_1^2) = (5+1)(5+2)/2 = 21$ and the components are given by

$$
\begin{aligned}
K^{11111} &= a_0^1 + 5a_1^1 x + 10a_2^1 x^2 + 10a_3^1 x^3 + 5a_4^1 x^4 + a_5^1 x^5, \\
K^{11112} &= (a_0^2 + 4a_1^2 x + 6a_2^2 x^2 + 4a_3^2 x^3 + b_4^1 x^4) \\
&\quad + t(a_1^1 + 4a_2^1 x + 6a_3^1 x^2 + 4a_4^1 x^3 + a_5^1 x^4), \\
K^{11122} &= (a_0^3 + 3a_1^3 x + 3b_2^2 x^2 + b_3^1 x^3) + 2t(a_1^2 + 3a_2^2 x + 3a_3^2 x^2 + b_4^1 x^3) \\
&\quad + t^2(a_2^1 + 3a_3^1 x + 3a_4^1 x^2 + a_5^1 x^3), \\
K^{11222} &= (b_0^3 + 3a_1^3 t + 3a_2^2 t^2 + a_3^1 t^3) + 2x(b_1^2 + 3b_2^2 t + 3a_3^2 t^2 + a_4^1 x^3) \\
&\quad + x^2(b_2^1 + 3b_3^1 t + 3b_4^1 t^2 + a_5^1 t^3), \\
K^{12222} &= (b_0^2 + 4b_1^2 t + 6b_2^2 t^2 + 4a_3^2 t^3 + a_4^1 t^4) \\
&\quad + x(b_1^1 + 4b_2^1 t + 6b_3^1 t^2 + 4b_4^1 t^3 + a_5^1 t^4), \\
K^{22222} &= b_0^1 + 5b_1^1 t + 10b_2^1 t^2 + 10b_3^1 t^3 + 5b_4^1 t^4 + a_5^1 t^5.
\end{aligned}
$$

$$(3.48)$$

In principle, based on the formulas (3.43), (3.44) and (3.45), we can now write down explicitly the general form of the elements of $\mathcal{K}^n(\mathbb{R}_1^2)$ for an arbitrary n, without any difficulty, just following the parameter scheme given above. To solve

Problem 1, we employ the MST-procedure [6] outlined in the previous section. Using the formulas (2.35), (3.43), (3.44) and (3.45), we arrive at the general formulas for the vector fields representing the infinitesimal action of the isometry group $I(\mathbb{R}_1^2)$ on the parameter space. As above, we have two cases corresponding to (3.44) and (3.45) respectively. **Case 1** $\quad n = 2k$

$$
\begin{aligned}
\mathbf{V}_1 \;=\; & a_1^1 \partial_{a_0^2} + a_2^1 \partial_{a_1^2} + \cdots + a_{n-1}^1 \partial_{a_{n-2}^2} \\
& + 2 a_1^2 \partial_{a_0^3} + 2 a_2^2 \partial_{a_1^3} + \cdots + 2 a_{n-3}^2 \partial_{a_{n-4}^3} \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& + \frac{n}{2} a_1^{\frac{n}{2}} \partial_{a_0^{\frac{n+2}{2}}} \\
& + \frac{n+2}{2} b_1^{\frac{n}{2}} \partial_{b_0^{\frac{n}{2}}} + \frac{n}{2} a_2^{\frac{n}{2}} \partial_{b_1^{\frac{n}{2}}} \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& + (n-1) b_1^2 \partial_{b_0^2} + (n-2) b_2^2 \partial_{b_1^2} + \cdots 2 b_{n-2}^2 \partial_{b_{n-3}^2} \\
& + n b_1^1 \partial_{b_0^1} + (n-1) b_2^1 \partial_{b_1^1} + \cdots + a_n^1 \partial_{b_{n-1}^1},
\end{aligned}
\tag{3.49}
$$

$$
\begin{aligned}
\mathbf{V}_2 \;=\; & b_1^1 \partial_{b_0^2} + b_2^1 \partial_{b_1^2} + \cdots + b_{n-1}^1 \partial_{a_{n-2}^2} \\
& + 2 b_1^2 \partial_{b_0^3} + 2 b_2^2 \partial_{b_1^3} + \cdots + 2 b_{n-3}^2 \partial_{a_{n-4}^3} \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& + \frac{n}{2} b_1^{\frac{n}{2}} \partial_{a_0^{\frac{n+2}{2}}} \\
& + \frac{n+2}{2} a_1^{\frac{n}{2}} \partial_{a_0^{\frac{n}{2}}} + \frac{n}{2} a_2^{\frac{n}{2}} \partial_{a_1^{\frac{n}{2}}} \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& + (n-1) a_1^2 \partial_{a_0^2} + (n-2) a_2^2 \partial_{a_1^2} + \cdots + 2 a_{n-2}^2 \partial_{a_{n-3}^2} \\
& + n a_1^1 \partial_{a_0^1} + (n-1) a_2^1 \partial_{a_1^1} + \cdots + a_n^1 \partial_{a_{n-1}^1},
\end{aligned}
\tag{3.50}
$$

$$
\begin{aligned}
\mathbf{V}_3 \;=\; & -na_0^2\partial_{a_0^1} - (n-1)a_1^2\partial_{a_1^1} - \cdots - 2a_{n-2}^2\partial_{a_{n-2}^1} - b_{n-1}^1\partial_{a_{n-1}^1} \\
& -[(n-1)a_0^3 + a_0^1]\partial_{a_0^2} - [(n-2)a_1^3 + a_1^1]\partial_{a_1^2} \\
& -\cdots - [2b_{n-3}^2 + a_{n-3}^1]\partial_{a_{n-3}^2} \\[4pt]
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\[4pt]
& -\frac{n}{2}[a_0^{\frac{n}{2}} + b_0^{\frac{n}{2}}]\partial_{a_0^{\frac{n+2}{2}}} - \cdots - [a_{n-2}^1 + b_{n-2}^1]\partial_{a_{n-2}^2} \\[4pt]
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\[4pt]
& -[(n-1)b_0^3 + b_0^1]\partial_{b_0^2} - \cdots - [2a_{n-3}^2 + b_{n-3}^1]\partial_{b_{n-3}^2} \\
& -nb_0^2\partial_{b_0^1} - (n-1)b_1^2\partial_{b_1^1} - \cdots - 2a_{n-2}^2\partial_{b_{n-2}^1} - a_{n-1}^1\partial_{b_{n-1}^1}.
\end{aligned}
\tag{3.51}
$$

Case 2 $n = 2k+1$

$$
\begin{aligned}
\mathbf{V}_1 \;=\; & a_1^1\partial_{a_0^2} + a_2^1\partial_{a_1^2} + \cdots + a_{n-1}^1\partial_{a_{n-2}^2} \\
& +2a_1^2\partial_{a_0^3} + 2a_2^2\partial_{a_1^3} + \cdots + 2a_{n-3}^2\partial_{a_{n-4}^3} \\[4pt]
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\[4pt]
& +\frac{n+1}{2}a_1^{\frac{n+1}{2}}\partial_{b_0^{\frac{n+1}{2}}} \\[4pt]
& +\frac{n+3}{2}b_1^{\frac{n-1}{2}}\partial_{b_0^{\frac{n-1}{2}}} + \frac{n+1}{2}b_1^{\frac{n-1}{2}}\partial_{b_1^{\frac{n-1}{2}}} + \frac{n-1}{2}a_3^{\frac{n-1}{2}}\partial_{b_2^{\frac{n-1}{2}}} \\[4pt]
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\[4pt]
& +(n-1)b_1^2\partial_{b_0^2} + (n-2)b_2^2\partial_{b_1^2} + \cdots 2a_{n-2}^2\partial_{b_{n-3}^2} \\
& +nb_1^1\partial_{b_0^1} + (n-1)b_2^1\partial_{b_1^1} + \cdots + 2b_{n-1}^1\partial_{b_{n-2}^1} + a_n^1\partial_{b_{n-1}^1},
\end{aligned}
\tag{3.52}
$$

$$
\begin{aligned}
\mathbf{V}_2 \;=\;& b_1^1 \partial_{b_0^2} + b_2^1 \partial_{b_1^2} + \cdots + b_{n-1}^1 \partial_{a_{n-2}^2} \\
& + 2b_1^2 \partial_{b_0^3} + 2b^2 \partial_{b_1^3} + \cdots + 2b_{n-3}^2 \partial_{a_{n-4}^3} \\[4pt]
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\[4pt]
& + \frac{n+1}{2} a_1^{\frac{n+1}{2}} \partial_{a_0^{\frac{n+1}{2}}} \\
& + \frac{n+3}{2} a_1^{\frac{n-1}{2}} \partial_{a_0^{\frac{n-1}{2}}} + \frac{n+1}{2} a_2^{\frac{n+1}{2}} \partial_{b_1^{\frac{n-1}{2}}} + \frac{n-1}{2} a_3^{\frac{n-1}{2}} \partial_{a_2^{\frac{n-1}{2}}} \\[4pt]
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\[4pt]
& + (n-1)a_1^2 \partial_{a_0^2} + (n-2)a_2^2 \partial_{a_1^2} + \cdots + 2a_{n-2}^2 \partial_{a_{n-3}^2} \\
& + n a_1^1 \partial_{a_0^1} + (n-1)a_2^1 \partial_{a_1^1} + \cdots + 2a_{n-1}^1 \partial_{a_{n-2}^1} + a_n^1 \partial_{a_{n-1}^1},
\end{aligned}
\tag{3.53}
$$

$$
\begin{aligned}
\mathbf{V}_3 \;=\;& -n a_0^2 \partial_{a_0^1} - (n-1)a_1^2 \partial_{a_1^1} - \cdots 2a_{n-2}^2 \partial_{a_{n-2}^1} - b_{n-1}^1 \partial_{a_{n-1}^1} \\
& - \left[(n-1)a_0^3 + a_0^1 \right] \partial_{a_0^2} - \left[(n-2)a_1^3 + a_1^1 \right] \partial_{a_1^2} \\
& - \cdots - \left[2b_{n-3}^2 + a_{n-3}^1 \right] \partial_{a_{n-3}^2} \\[4pt]
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\[4pt]
& - \frac{n-1}{2} [a_1^{\frac{n-1}{2}} + b_1^{\frac{n-1}{2}}] \partial_{a_1^{\frac{n+1}{2}}} - \cdots - [a_{n-2}^1 + b_{n-2}^1] \partial_{a_{n-2}^2} \\[4pt]
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\[4pt]
& - [(n-1)b_0^3 + b_0^1] \partial_{b_0^2} - \cdots - [2a_{n-3}^2 + b_{n-3}^1] \partial_{b_{n-3}^2} \\
& - n b_0^2 \partial_{b_0^1} - (n-1)b_1^2 \partial_{b_1^1} - \cdots - 2a_{n-2}^2 \partial_{b_{n-2}^1} - a_{n-1}^1 \partial_{b_{n-1}^1}.
\end{aligned}
\tag{3.54}
$$

We remark that in both cases the vector fields $\mathbf{V}_1$, $\mathbf{V}_2$ and $\mathbf{V}_3$ correspond to the generators $\mathbf{T}$, $\mathbf{X}$ and $\mathbf{H}$ given by (2.22) respectively. Moreover, it is easy to verify directly that the vector fields $-\mathbf{V}_1$, $-\mathbf{V}_2$ and $-\mathbf{V}_3$ satisfy the same commutator relations (2.23) as $\mathbf{T}$, $\mathbf{X}$ and $\mathbf{H}$. We conclude therefore that $\mathbf{V}_i$, $i = 1, 2, 3$ represent the infinitesimal action of the isometry group $I(\mathbb{R}_1^2)$ on the parameter space Σ defined by $\mathcal{K}^n(\mathbb{R}_1^2)$ for each $n \geq 1$ and we have proven the ITKT analogue of Lemma 1.1 of Cayley [20]:

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Lemma 3.1 *The action of the isometry group $I(\mathbb{R}_1^2)$ on the vector space $\mathcal{K}^n(\mathbb{R}_1^2)$ has the infinitesimal generators (3.49), (3.50) and (3.51) when n is even and (3.52), (3.53) and (3.54) when n is odd.*

Example 3.3 Consider the vector space $\mathcal{K}^4(\mathbb{R}_1^2)$. Using the formulas (3.47), (3.49), (3.50) and (3.51), we derive the three vector fields representing the infinitesimal action of the isometry group $I(\mathbb{R}_1^2)$ on the vector space $\mathcal{K}^4(\mathbb{R}_1^2)$.

$$
\begin{aligned}
\mathbf{V}_1 \;=\; & a_1^1\partial_{a_0^2} + a_2^1\partial_{a_1^2} + a_3^1\partial_{a_2^2} \\
& +2a_1^2\partial_{a_0^3} \\
& +3b_1^2\partial_{b_0^2} + 2a_2^2\partial_{b_1^2} \\
& +4b_1^1\partial_{b_0^1} + 3b_2^1\partial_{b_1^1} + 2b_3^1\partial_{b_2^1} + a_4^1\partial_{b_3^1},
\end{aligned}
\tag{3.55}
$$

$$
\begin{aligned}
\mathbf{V}_2 \;=\; & b_1^1\partial_{b_0^2} + b_2^1\partial_{b_1^2} + b_3^1\partial_{a_2^2} \\
& +2b_1^2\partial_{a_0^3} \\
& +3a_1^2\partial_{a_0^2} + 2a_2^2\partial_{a_1^2} \\
& +4a_1^1\partial_{b_0^1} + 3a_2^1\partial_{b_1^1} + 2a_3^1\partial_{b_2^1} + a_4^1\partial_{a_3^1},
\end{aligned}
\tag{3.56}
$$

$$
\begin{aligned}
\mathbf{V}_3 \;=\; & -4a_0^2\partial_{a_0^1} - 3a_1^2\partial_{a_1^1} - 2a_2^2\partial_{a_2^1} - b_3^1\partial_{a_3^1} \\
& -(3a_0^3 + a_0^1)\partial_{a_0^2} - (2b_1^2 + a_1^1)\partial_{a_1^2} \\
& -2(a_0^2 + b_0^2)\partial_{a_0^3} - (a_2^1 + b_2^1)\partial_{a_2^2} \\
& -(3a_0^3 + b_0^1)\partial_{b_0^2} - (2a_1^2 + b_1^1)\partial_{b_1^2} \\
& -4b_0^2\partial_{b_0^1} - 3b_1^2\partial_{b_1^1} - 2a_2^2\partial_{b_2^1} - a_3^1\partial_{b_3^1}.
\end{aligned}
\tag{3.57}
$$

Example 3.4 Consider the vector space $\mathcal{K}^5(\mathbb{R}_1^2)$. Using the formulas (3.48) and (3.52), (3.53) and (3.54), we derive the three vector fields representing the in-

finitesimal action of the isometry group $I(\mathbb{R}_1^2)$ on the vector space $\mathcal{K}^5(\mathbb{R}_1^2)$.

$$
\begin{aligned}
\mathbf{V}_1 \;=\; & a^1_1 \partial_{a^2_0} + a^1_2 \partial_{a^2_1} + a^1_3 \partial_{a^2_2} + a^1_4 \partial_{a^2_3} \\
& + 2a^2_1 \partial_{a^3_0} + 2a^2_2 \partial_{a^3_1} \\
& + 3a^3_1 \partial_{b^3_0} \\
& + 4b^2_1 \partial_{b^2_0} + 3b^2_2 \partial_{b^2_1} + 2a^2_3 \partial_{b^2_2} \\
& + 5b^1_1 \partial_{b^1_0} + 4b^1_2 \partial_{b^1_1} + 3b^1_3 \partial_{b^1_2} + 2b^1_4 \partial_{b^1_3} + a^1_5 \partial_{b^1_4},
\end{aligned}
\tag{3.58}
$$

$$
\begin{aligned}
\mathbf{V}_2 \;=\; & b^1_1 \partial_{b^2_0} + b^1_2 \partial_{b^2_1} + b^1_3 \partial_{b^2_2} + b^1_4 \partial_{a^2_3} \\
& + 2b^2_1 \partial_{b^3_0} + 2a^2_2 \partial_{a^3_1} \\
& + 3a^3_1 \partial_{a^3_0} \\
& + 4a^2_1 \partial_{a^2_0} + 3a^2_2 \partial_{a^2_1} + 2a^2_3 \partial_{a^2_2} \\
& + 5a^1_1 \partial_{a^1_0} + 4a^1_2 \partial_{a^1_1} + 3a^1_3 \partial_{a^1_2} + 2a^1_4 \partial_{a^1_3} + a^1_5 \partial_{a^1_4},
\end{aligned}
\tag{3.59}
$$

$$
\begin{aligned}
\mathbf{V}_3 \;=\; & -5a^2_0 \partial_{a^1_0} - 4a^2_1 \partial_{a^1_1} - 3a^2_2 \partial_{a^1_2} - 2a^2_3 \partial_{a^1_3} - b^1_4 \partial_{a^1_4} \\
& - (4a^3_0 + a^1_0) \partial_{a^2_0} - (3a^3_1 + a^1_1) \partial_{a^2_1} - (2b^2_2 + a^1_2) \partial_{a^2_2} \\
& - (3b^3_0 + 2a^2_0) \partial_{a^3_0} - 2(b^2_1 + a^2_1) \partial_{a^3_1} - (a^1_3 + b^1_3) \partial_{a^2_3} \\
& - (3a^3_0 + 2b^2_0) \partial_{b^3_0} \\
& - (4b^3_0 + b^1_0) \partial_{b^2_0} - (3a^3_1 + b^1_1) \partial_{b^2_1} - (2a^2_2 + b^1_2) \partial_{b^2_2} \\
& - 5b^2_0 \partial_{b^1_0} - 4b^2_1 \partial_{b^1_1} - 3b^2_2 \partial_{b^1_2} - 2a^2_3 \partial_{b^1_3} - a^1_4 \partial_{b^1_4}.
\end{aligned}
\tag{3.60}
$$

Using the result of Lemma 3.1 we can now employ the infinitesimal generators to compute the $I(\mathbb{R}_1^2)$-invariants.

Proposition 3.1 *A function $I : \Sigma \to \mathbb{R}$ is an $I(\mathbb{R}_1^2)$-invariant of the induced action of the isometry group $I(\mathbb{R}_1^2)$ on the vector space $\mathcal{K}^n(\mathbb{R}_1^2)$ for a specific $n \geq 1$ if and only if it satisfies the infinitesimal criteria*

$$
\mathbf{V}_1(I) = \mathbf{V}_2(I) = \mathbf{V}_3(I) = 0,
\tag{3.61}
$$

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where Σ is the parameter space of $\mathcal{K}^n(\mathbb{R}^2_1)$ and the vector fields $\mathbf{V}_i$, $i = 1, 2, 3$ are given by (3.49), (3.50) and (3.51) when n is even and (3.52), (3.53) and (3.54) when n is odd.

Corollary 3.1 *For a given $n \geq 1$ the parameter a^1_n (refer to the formulas (3.44) and (3.45) when n is even and odd respectively) is a fundamental $I(\mathbb{R}^2_1)$-invariant of the vector space $\mathcal{K}^n(\mathbb{R}^2_1)$.*

Proof. Follows from Proposition 3.1 and the formulas (3.49), (3.50) and (3.51) when n is even and (3.52), (3.53) and (3.54) when n is odd. $\square$ In

view of Proposition 3.1, the problem of the determination of the space of $I(\mathbb{R}^2_1)$-invariants reduces to solving the system of linear PDEs (3.61). For larger values of n the problem becomes very challenging computationally. The method of characteristics may fail, in which case one can employ the method of undetermined coefficients in conjuncture with the result of Theorem 1.1, as well as computer algebra. This technique was used with a remarkable success in Horwood *et al* [7] to solve the problem of the determination of the space of $I(\mathbb{R}^3)$-invariants of the vector space $\mathcal{K}^2(\mathbb{R}^3)$, where $\mathbb{R}^3$ denotes the Euclidean space. The concept of a *covariant* in ITKT was introduced in [9]. Proposition 3.1 entails the corresponding criterion for $I(\mathbb{R}^2_1)$-covariants of the vector spaces $\mathcal{K}^n(\mathbb{R}^2_1)$, $n \geq 1$.

Theorem 3.1 *Let $\mathcal{K}^n(\mathbb{R}^2_1)$ be the vector space of Killing tensors of valence n defined in the Minkowski plane $\mathbb{R}^2_1$ for a fixed $n \geq 1$. A function $C : \Sigma \times \mathbb{R}^2_1 \to \mathbb{R}$ is an $I(\mathbb{R}^2_1)$-covariant of $\mathcal{K}^n(\mathbb{R}^2_1)$ if and only if it satisfies the infinitesimal invariance conditions*

$$\tilde{\mathbf{V}}_1(C) = \tilde{\mathbf{V}}_2(C) = \tilde{\mathbf{V}}_3(C) = 0, \tag{3.62}$$

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where the infinitesimal generators are

$$
\begin{aligned}
\tilde{\mathbf{V}}_1 &= \mathbf{V}_1 + \partial_t, \\
\tilde{\mathbf{V}}_2 &= \mathbf{V}_2 + \partial_x, \\
\tilde{\mathbf{V}}_3 &= \mathbf{V}_3 + x\partial_t + t\partial_x,
\end{aligned}
\tag{3.63}
$$

Σ *is the parameter space of* $\mathcal{K}^n(\mathbb{R}_1^2)$ *and the vector fields* $\mathbf{V}_i$, $i = 1, 2, 3$ *are given by (3.49), (3.50) and (3.51) when* n *is even and (3.52), (3.53) and (3.54) when* n *is odd.*

4 Conclusions

After all, in this paper we have formulated and proven only *an* ITKT analogue of Cayley's Lemma in CIT. A similar result for the vector spaces $\mathcal{K}^n(\mathbb{R}^2)$, $n \geq 1$ (here $\mathbb{R}^2$ denotes the Euclidean plane) can be obtained *mutatis mutandis*. Indeed, it is obvious that the corresponding formulas will differ only by signs. More challenging problems are to extend the result to two-dimensional spaces of non-zero curvature, namely when the underlying manifold is $\mathbb{S}^2$ (two-sphere) or $\mathbb{H}^2$ (hyperbolic plane). The work in this direction is underway.

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